The principle of local reflexivity for operator ideals and its implications

Frank Oertel
Zurich Financial Services, Reinsurance
Corporate Development and Risk Management
General Guisan-Quai 26
8022 Zurich, Switzerland
E-mail address: oertel@e-math.ams.org

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Abstract

We present a survey of past research activities and current results in constructing a mathematical framework describing the principle of local reflexivity for operator ideals and reveal further applications involving operator ideal products consisting of operators which factor through a Hilbert space.

1 Introduction

This survey paper is devoted to the presentation of the author‘s research results related to the investigation of operator ideals $(\mathfrak{A}, \mathbf{A})$ which allow a transfer of the norm estimation in the classical principle of local reflexivity to their ideal (quasi–)norm $\mathbf{A}$. This research originated from the objective to facilitate the search for a non–accessible maximal normed Banach ideal (which is the same as a non–accessible finitely generated tensor norm in the sense of Grothendieck) and lead to the dissertation [9] in 1990. Later, in 1993, Pisier constructed such a counterexample (cf. [1, 31.6.]). Since each right–accessible maximal Banach ideal $(\mathfrak{A}, \mathbf{A})$ even satisfies such a *principle of local reflexivity for operator ideals* (called $\mathfrak{A}$–*LRP*), Pisier‘s counterexample of a non–accessible maximal Banach ideal naturally lead to the search for counterexamples of maximal Banach ideals $(\mathfrak{A}_0, \mathbf{A}_0)$ which even do not satisfy the $\mathfrak{A}_0$-LRP, implying surprising relations between the existence of a norm on product operator ideals of type $\mathfrak{B}\circ\mathfrak{L}_2$ (where $\mathfrak{L}_2$ denotes the class of all operators which factor through a Hilbert space), the extension of finite rank operators with respect to a suitable operator ideal norm and the principle of local reflexivity for operator ideals (cf. [13]). The basic objects, connecting these different aspects, are product operator ideals with property (I) and property (S), introduced by Jarchow and Ott (see [7]). In the widest sense, a product operator ideal $\mathfrak{A} \circ \mathfrak{B}$ has the property (I), if

$$(\mathfrak{A} \circ \mathfrak{B}) \cap \mathfrak{F} = (\mathfrak{A} \cap \mathfrak{F}) \circ \mathfrak{B}$$

and the property (S), if

$$(\mathfrak{A} \circ \mathfrak{B}) \cap \mathfrak{F} = \mathfrak{A} \circ (\mathfrak{B} \cap \mathfrak{F})$$

(where $\mathfrak{F}$ denotes the class of all finite rank operators) so that each finite rank operator in $\mathfrak{A} \circ \mathfrak{B}$ is the composition of two operators, one of which is of finite rank. Since each operator ideal which contains $\mathfrak{L}_2$ as a factor, has both, the property (I) and the property (S), Hilbert space factorization crystallized out as a fundamental key in these investigations.

2 The framework

In this section, we introduce the basic notation and terminology which we will use throughout in this paper. We only deal with Banach spaces and most of our notations and definitions concerning Banach spaces and operator ideals are standard. We refer the reader to the monographs [1], [2] and [14] for the necessary background in operator ideal theory and the related terminology. Infinite dimensional Banach spaces over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ are denoted throughout by W, X, Y and Z in contrast to the letters E, F and G which are used for finite dimensional Banach spaces only. The space of all operators (continuous linear maps) from X to Y is denoted by $\mathfrak{L}$(X,Y), and for the identity operator on X, we write Id_X. The collection of all finite rank (resp. approximable) operators from X to Y is denoted by $\mathfrak{F}$(X,Y) (resp. $\overline{\mathfrak{F}}$(X,Y)), and $\mathfrak{E}$(X,Y) indicates the collection of all operators, acting between finite dimensional Banach spaces X and Y (elementary operators). The dual of a Banach space X is denoted by X', and X'' denotes its bidual $(X')'$. If $T \in \mathfrak{L}$(X,Y) is an operator, we indicate that it is a metric injection by writing $T : X \overset{1}{\hookrightarrow} Y$, and if it is a metric surjection, we write $T : X \overset{1}{\twoheadrightarrow} Y$. If X is a Banach space, E a finite dimensional subspace of X and K a finite codimensional subspace of X, then $B_X := \{x \in X : \|x\| \leq 1\}$ denotes the closed unit ball, $J_E^X : E \overset{1}{\hookrightarrow} X$ the canonical metric injection and $Q_K^X : X \overset{1}{\twoheadrightarrow} X/K$ the canonical metric surjection. Finally, $T' \in \mathfrak{L}(Y', X')$ denotes the dual operator of $T \in \mathfrak{L}$(X,Y).

If $(\mathfrak{A}, \mathbf{A})$ and $(\mathfrak{B}, \mathbf{B})$ are given quasi–Banach ideals, we will use throughout the shorter notation $(\mathfrak{A}^d, \mathbf{A}^d)$ for the dual ideal and the abbreviation $\mathfrak{A} \overset{1}{=} \mathfrak{B}$ for the isometric equality $(\mathfrak{A}, \mathbf{A}) = (\mathfrak{B}, \mathbf{B})$. We write $\mathfrak{A} \subseteq \mathfrak{B}$ if, regardless of the Banach spaces X and Y, we have $\mathfrak{A}(X, Y) \subseteq \mathfrak{B}(X, Y)$. If X_0 is a fixed Banach space, we write $\mathfrak{A}(X_0, \cdot) \subseteq \mathfrak{B}(X_0, \cdot)$ (resp. $\mathfrak{A}(\cdot, X_0) \subseteq \mathfrak{B}(\cdot, X_0)$) if, regardless of the Banach space Z we have $\mathfrak{A}(X_0, Z) \subseteq \mathfrak{B}(X_0, Z)$ (resp. $\mathfrak{A}(Z, X_0) \subseteq \mathfrak{B}(Z, X_0)$). The metric inclusion $(\mathfrak{A}, \mathbf{A}) \subseteq (\mathfrak{B}, \mathbf{B})$ is often shortened by $\mathfrak{A} \overset{1}{\subseteq} \mathfrak{B}$. If $\mathbf{B}(T) \leq \mathbf{A}(T)$ for all finite rank (resp. elementary) operators $T \in \mathfrak{F}$ (resp. $T \in \mathfrak{E}$), we sometimes use the abbreviation $\mathfrak{A} \overset{\mathfrak{F}}{\subseteq} \mathfrak{B}$ (resp. $\mathfrak{A} \overset{\mathfrak{E}}{\subseteq} \mathfrak{B}$).

First we recall the basic notions of Grothendieck's metric theory of tensor products (cf., eg., [1], [3], [5], [8]), which together with Pietsch's theory of operator ideals spans the mathematical frame of this paper. A *tensor norm* α is a mapping which assigns to each pair (X, Y) of Banach spaces a norm $\alpha(\cdot; X, Y)$ on the algebraic tensor product $X \otimes Y$ (shorthand: $X \otimes_\alpha Y$ and $X \tilde{\otimes}_\alpha Y$ for the completion) so that

- $\varepsilon \leq \alpha \leq \pi$
- α satisfies the metric mapping property: If $S \in \mathfrak{L}(X, Z)$ and $T \in \mathfrak{L}(Y, W)$, then $\|S \otimes T : X \otimes_\alpha Y \longrightarrow Z \otimes_\alpha W\| \leq \|S\| \, \|T\|$.

Wellknown examples are the injective tensor norm ε, which is the smallest one, and the projective tensor norm π, which is the largest one. For other important examples we refer

to [1], [3], or [8]. Each tensor norm α can be extended in two natural ways. For this, denote for given Banach spaces X and Y

$$\mathrm{FIN}(X) := \{E \subseteq X \mid E \in \mathrm{FIN}\} \text{ and } \mathrm{COFIN}(X) := \{L \subseteq X \mid X/L \in \mathrm{FIN}\},$$

where FIN stands for the class of all finite dimensional Banach spaces. Let $z \in X \otimes Y$. Then the *finite hull* $\overrightarrow{\alpha}$ is given by

$$\overrightarrow{\alpha}(z; X, Y) := \inf\{\alpha(z; E, F) \mid E \in \mathrm{FIN}(X),\ F \in \mathrm{FIN}(Y),\ z \in E \otimes F\},$$

and the *cofinite hull* $\overleftarrow{\alpha}$ of α is given by

$$\overleftarrow{\alpha}(z; X, Y) := \sup\{\alpha(Q_K^X \otimes Q_L^Y(z); X/K, Y/L) \mid K \in \mathrm{COFIN}(X),\ L \in \mathrm{COFIN}(Y)\}.$$

α is called *finitely generated* if $\alpha = \overrightarrow{\alpha}$, *cofinitely generated* if $\alpha = \overleftarrow{\alpha}$ (it is always true that $\overleftarrow{\alpha} \leq \alpha \leq \overrightarrow{\alpha}$). α is called *right–accessible* if $\overleftarrow{\alpha}(z; E, Y) = \overrightarrow{\alpha}(z; E, Y)$ for all $(E, Y) \in \mathrm{FIN} \times \mathrm{BAN}$, *left–accessible* if $\overleftarrow{\alpha}(z; X, F) = \overrightarrow{\alpha}(z; X, F)$ for all $(X, F) \in \mathrm{BAN} \times \mathrm{FIN}$, and *accessible* if it is right–accessible and left–accessible. α is called *totally accessible* if $\overleftarrow{\alpha} = \overrightarrow{\alpha}$. The injective norm ε is totally accessible, the projective norm π is accessible – but not totally accessible, and Pisier's construction implies the existence of a finitely generated tensor norm which is neither left– nor right–accessible (see [1, 31.6.]).

There exists a powerful one–to–one correspondence between finitely generated tensor norms and maximal Banach ideals which links thinking in terms of operators with "tensorial" thinking and which allows to transfer notions in the "tensor language" to the "operator language" and conversely. In particular, this one–to–one correspondence helps to extend the trace duality

$$< S, T >:= tr(TS) \ \ (S \in \mathfrak{F}(X, Y),\ T \in \mathfrak{F}(Y, X))$$

to operator ideals by using tensor product methods. We refer the reader to [1] and [9] for detailed informations concerning this subject. Let X, Y be Banach spaces and $z = \sum_{i=1}^{n} x_i' \otimes y_i$ be an Element in $X' \otimes Y$. Then $T_z(x) := \sum_{i=1}^{n} \langle x, x_i' \rangle\ y_i$ defines a finite rank operator $T_z \in \mathfrak{F}(X, Y)$ which is independent of the representation of z in $X' \otimes Y$. Let α be a finitely generated tensor norm and $(\mathfrak{A}, \mathbf{A})$ be a maximal Banach ideal. α and $(\mathfrak{A}, \mathbf{A})$ are said to be *associated*, notation:

$$(\mathfrak{A}, \mathbf{A}) \sim \alpha \text{ (shorthand: } \mathfrak{A} \sim \alpha, \text{ resp. } \alpha \sim \mathfrak{A}),$$

if for all $E, F \in \mathrm{FIN}$

$$\mathfrak{A}(E, F) = E' \otimes_\alpha F$$

holds isometrically: $\mathbf{A}(T_z) = \alpha(z; E', F)$.

Since we will use them throughout in this paper, let us recall the important notions of the conjugate operator ideal (cf. [4], [7] and [10]) and the adjoint operator ideal (all details can be found in the standard references [1] and [14]). Let $(\mathfrak{A}, \mathbf{A})$ be a quasi–Banach ideal.

- Let $\mathfrak{A}^\Delta(X, Y)$ be the set of all $T \in \mathfrak{L}(X, Y)$ which satisfy

$$\mathbf{A}^\Delta(T) := \sup\{| \, tr(TL) \, | \ \ | \, L \in \mathfrak{F}(Y, X), \mathbf{A}(L) \leq 1\} < \infty.$$

  Then a Banach ideal $(\mathfrak{A}^\Delta, \mathbf{A}^\Delta)$ is obtained (here, $tr(\cdot)$ denotes the usual trace for finite rank operators). It is called the *conjugate ideal* of $(\mathfrak{A}, \mathbf{A})$.

- Let $\mathfrak{A}^*(X,Y)$ be the set of all $T \in \mathfrak{L}(X,Y)$ which satisfy

$$\mathbf{A}^*(T) := \sup\{|\, tr(TJ_E^X S Q_K^Y\} \,| \quad | \, E \in \text{FIN}(X), K \in \text{COFIN}(Y), \mathbf{A}(S) \leq 1\} < \infty.$$

  Then a Banach ideal $(\mathfrak{A}^*, \mathbf{A}^*)$ is obtained. It is called the *adjoint operator ideal* of $(\mathfrak{A}, \mathbf{A})$.

By definition, it immediately follows that $\mathfrak{A}^\Delta \overset{1}{\subseteq} \mathfrak{A}^*$. Another easy, yet important observation is the following: let $(\mathfrak{A}, \mathbf{A})$ be a quasi–Banach ideal and $(\mathfrak{B}, \mathbf{B})$ be a quasi–Banach ideal. If $\mathfrak{A} \overset{\mathfrak{E}}{\subseteq} \mathfrak{B}$, then $\mathfrak{B}^* \overset{1}{\subseteq} \mathfrak{A}^*$, and $\mathfrak{A} \overset{\mathfrak{F}}{\subseteq} \mathfrak{B}$ implies the inclusion $\mathfrak{B}^\Delta \overset{1}{\subseteq} \mathfrak{A}^\Delta$. In particular, it follows that $\mathfrak{A}^{\Delta *} \overset{1}{=} \mathfrak{A}^{**}$ and $(\mathfrak{A}^{\Delta\Delta})^* \overset{1}{=} \mathfrak{A}^*$. A deeper investigation of relations between the Banach ideals $(\mathfrak{A}^\Delta, \mathbf{A}^\Delta)$ and $(\mathfrak{A}^*, \mathbf{A}^*)$ needs the help of an important local property, known as accessibility, which can be viewed as a local version of injectivity and surjectivity. All necesary details about accessibility of operator ideals and its applications can be found in [1], [10], [11] and [12]. So let us recall :

- A quasi–Banach ideal $(\mathfrak{A}, \mathbf{A})$ is called *right–accessible*, if for all $(E, Y) \in \text{FIN} \times \text{BAN}$, operators $T \in \mathfrak{L}(E, Y)$ and $\varepsilon > 0$ there are $F \in \text{FIN}(Y)$ and $S \in \mathfrak{L}(E, F)$ so that $T = J_F^Y S$ and $\mathbf{A}(S) \leq (1+\varepsilon)\mathbf{A}(T)$.

- $(\mathfrak{A}, \mathbf{A})$ is called *left–accessible*, if for all $(X, F) \in \text{BAN} \times \text{FIN}$, operators $T \in \mathfrak{L}(X, F)$ and $\varepsilon > 0$ there are $L \in \text{COFIN}(X)$ and $S \in \mathfrak{L}(X/L, F)$ so that $T = SQ_L^X$ and $\mathbf{A}(S) \leq (1+\varepsilon)\mathbf{A}(T)$.

- A left–accessible and right–accessible quasi–Banach ideal is called *accessible.*

- $(\mathfrak{A}, \mathbf{A})$ is *totally accessible*, if for every finite rank operator $T \in \mathfrak{F}(X, Y)$ acting between Banach spaces X, Y and $\varepsilon > 0$ there are $(L, F) \in \text{COFIN}(X) \times \text{FIN}(Y)$ and $S \in \mathfrak{L}(X/L, F)$ so that $T = J_F^Y S Q_L^X$ and $\mathbf{A}(S) \leq (1+\varepsilon)\mathbf{A}(T)$.

Given quasi–Banach ideals $(\mathfrak{A}, \mathbf{A})$ and $(\mathfrak{B}, \mathbf{B})$, let $(\mathfrak{A} \circ \mathfrak{B}, \mathbf{A} \circ \mathbf{B})$ be the corresponding product ideal and $(\mathfrak{A} \circ \mathfrak{B}^{-1}, \mathbf{A} \circ \mathbf{B}^{-1})$ (resp. $(\mathfrak{A}^{-1} \circ \mathfrak{B}, \mathbf{A}^{-1} \circ \mathbf{B})$) the corresponding "right–quotient" (resp. "left–quotient"). We write $(\mathfrak{A}^{inj}, \mathbf{A}^{inj})$, to denote the *injective hull* of $\mathfrak{A}$, the unique smallest injective quasi–Banach ideal which contains $(\mathfrak{A}, \mathbf{A})$, and $(\mathfrak{A}^{sur}, \mathbf{A}^{sur})$, the *surjective hull* of $\mathfrak{A}$, is the unique smallest surjective quasi–Banach ideal which contains $(\mathfrak{A}, \mathbf{A})$. Of particular importance are the quotients $\mathfrak{A}^{\dashv} := \mathfrak{I} \circ \mathfrak{A}^{-1}$ and $\mathfrak{A}^{\vdash} := \mathfrak{A}^{-1} \circ \mathfrak{I}$ and their relations to $\mathfrak{A}^\Delta$ and $\mathfrak{A}^*$, treated in detail in [9] and [12].

In addition to the maximal Banach ideal $(\mathfrak{L}, \|\cdot\|) \sim \varepsilon$ we mainly will be concerned with the maximal Banach ideals $(\mathfrak{I}, \mathbf{I}) \sim \pi$ (integral operators), $(\mathfrak{L}_2, \mathbf{L}_2) \sim w_2$ (Hilbertian operators), $(\mathfrak{D}_2, \mathbf{D}_2) \overset{1}{=} (\mathfrak{L}_2^*, \mathbf{L}_2^*) \overset{1}{=} \mathfrak{P}_2^d \circ \mathfrak{P}_2 \sim w_2^*$ (2–dominated operators), $(\mathfrak{P}_p, \mathbf{P}_p) \sim g_p\backslash = g_q^*$ (absolutely p–summing operators), $1 \leq p \leq \infty, \frac{1}{p} + \frac{1}{q} = 1$, $(\mathfrak{L}_\infty, \mathbf{L}_\infty) \overset{1}{=} (\mathfrak{A}_1^*, \mathbf{P}_1^*) \sim w_\infty$ and $(\mathfrak{L}_1, \mathbf{L}_1) \overset{1}{=} (\mathfrak{P}_1^{*d}, \mathbf{P}_1^{*d}) \sim w_1$. We also consider the maximal Banach ideals $(\mathfrak{C}_2, \mathbf{C}_2) \sim c_2$ (cotype 2 operators) and $(\mathfrak{A}_P, \mathbf{A}_P) \sim \alpha_P$ (Pisier‘s counterexample of a maximal Banach ideal which is neither right– nor left–accessible (cf. [1], 31.6)).

3 The principle of local reflexivity for operator ideals

Let $(\mathfrak{A}, \mathbf{A})$ be a *maximal* Banach ideal. Then, $\mathfrak{A}^\Delta$ always is right–accessible (cf. [12]). The natural question whether $\mathfrak{A}^\Delta$ is *left*–accessible is still open[1] and leads to interesting and non–trivial results concerning the local structure of $\mathfrak{A}^\Delta$. Deeper investigations of the left–accessibility of $\mathfrak{A}^\Delta$ namely lead to a link with a principle of local reflexivity for operator ideals (a detailed discussion can be found in [9] and [10]) which allows a transmission of the operator norm estimation in the classical principle of local reflexivity to the ideal norm $\mathbf{A}$. So let us recall the

Definition 3.1 *Let E and Y be Banach spaces, E finite dimensional, $F \in FIN(Y')$ and $T \in \mathfrak{L}(E, Y'')$. Let $(\mathfrak{A}, \mathbf{A})$ be a quasi–Banach ideal and $\epsilon > 0$. We say that the principle of $\mathfrak{A}-$local reflexivity (short: $\mathfrak{A} - LRP$) is satisfied, if there exists an operator $S \in \mathfrak{L}(E, Y)$ so that*

(1) $\mathbf{A}(S) \leq (1 + \epsilon) \cdot \mathbf{A}^{**}(T)$

(2) $\langle Sx, y' \rangle = \langle y', Tx \rangle$ *for all* $(x, y') \in E \times F$

(3) $j_Y Sx = Tx$ *for all* $x \in T^{-1}(j_Y(Y))$.

Although both, the quasi–Banach ideal $\mathfrak{A}$ and the 1–Banach ideal $\mathfrak{A}^{**}$ are involved, the unbalance can be justified by the following statement which holds for arbitrary quasi–Banach ideals (see [10]):

Theorem 3.1 *Let $(\mathfrak{A}, \mathbf{A})$ be a quasi–Banach ideal. Then the following statements are equivalent:*

(i) $\mathfrak{A}^\Delta$ *is left–accessible*

(ii) $\mathfrak{A}^{**}(E, Y'') \stackrel{1}{=} \mathfrak{A}(E, Y)''$ *for all* $(E, Y) \in FIN \times BAN$

(iii) The $\mathfrak{A} - LRP$ *holds.*

If we only assume that $(\mathfrak{A}, \mathbf{A})$ is a maximal Banach ideal, the previous statement is contained in the following important observation (cf. [9]):

Theorem 3.2 *Let $(\mathfrak{A}, \mathbf{A}) \sim \alpha$ be an arbitrary maximal Banach ideal. Then the following statements are equivalent:*

(i) $\mathfrak{A}^\Delta$ *is left–accessible*

(ii) $\mathfrak{A}(E, Y'') \stackrel{1}{=} \mathfrak{A}(E, Y)''$ *for all* $(E, Y) \in FIN \times BAN$

(iii) $(\mathfrak{A}^d)^\Delta(X_0, Y') \stackrel{1}{=} (X_0 \otimes_{\overleftarrow{\alpha}} Y)'$ *for all Banach spaces X_0 with the metric approximation property and all $Y \in BAN$*

(iv) The $\mathfrak{A} - LRP$ *holds.*

[1]For minimal Banach ideals $(\mathfrak{A}, \mathbf{A})$, there exist counterexamples: the conjugate of $\mathfrak{A}_P^{\min}$ neither is right–accessible nor left–accessible (cf. [10]).

One reason which leads to extreme persistent difficulties concerning the verification of the $\mathfrak{A}-LRP$ for a given maximal Banach ideal $\mathfrak{A}$, is the behaviour of the bidual $(\mathfrak{A}^{\Delta})^{dd}$: although we know that in general $(\mathfrak{A}^{\Delta})^{dd}$ and $(\mathfrak{A}^{\Delta})^{d}$ both are accessible (see [9] and [10]) and that $(\mathfrak{A}^{\Delta})^{dd} \overset{1}{\subseteq} \mathfrak{A}^{\Delta}$, we do not know whether $\mathfrak{A}^{\Delta}(X,Y)$ and $(\mathfrak{A}^{\Delta})^{dd}(X,Y)$ coincide *isometrically* for *all* Banach spaces X and Y. If we allow in addition the approximation property of X or Y, then we may state the following

Theorem 3.3 *Let $(\mathfrak{A}, \mathbf{A})$ be an arbitrary maximal Banach ideal and X, Y be arbitrary Banach spaces. Then*

$$\mathfrak{A}^{d\Delta}(X,Y) \overset{1}{=} \mathfrak{A}^{\Delta d}(X,Y)$$

holds in each of the following two cases:

(i) X' has the metric approximation property

(ii) Y' has the metric approximation property and the $\mathfrak{A}^{d} - LRP$ is satisfied.

Which class of operator ideals $(\mathfrak{A}, \mathbf{A})$ does satisfy the $\mathfrak{A} - LRP$? At least, all maximal and right–accessible Banach ideals belong to this class (cf. [13]) due to the following

Theorem 3.4 *Let $(\mathfrak{A}, \mathbf{A})$ be an arbitrary Banach ideal. If $\mathfrak{A}$ is right–accessible and ultra-stable, then the $\mathfrak{A} - LRP$ is satisfied.*

Pisier‘s counterexample of the maximal Banach ideal $(\mathfrak{A}_P, \mathbf{A}_P)$ which neither is left–accessible nor right–accessible (cf. [1], 31.6) implies that in particular $(\mathfrak{A}_P^*, \mathbf{A}_P^*)$ neither is left–accessible nor right–accessible. Thinking at $\mathfrak{A}_P^{\Delta} \overset{1}{\subseteq} \mathfrak{A}_P^*$, this leads to the natural and even more tough question whether the $\mathfrak{A}_P - LRP$ is true or false. Since the Pisier space P does not have the approximation property, $\mathfrak{A}_P^{\Delta}$ cannot be totally accessible. Is it even true that $(\mathfrak{A}_P^{\Delta})^{inj} \overset{1}{=} \mathfrak{P}_1 \circ (\mathfrak{A}_P)^{-1}$ is not totally accessible? If this is the case, the $\mathfrak{A}_P - LRP$ will be false. However, $\mathfrak{A}_P^{\Delta}$ is not injective (cf. [13]). What about the left accessibility of $\mathfrak{A}_P^{*\Delta}$? Can we mimic Pisier‘s proof to construct a similar counterexample of a maximal Banach ideal $(\mathfrak{A}_0, \mathbf{A}_0)$ which even does not satisfy the $\mathfrak{A}_0 - LRP$ (cf. [1, 31.6.])? Unfortunately, the proof of the following statement leads to a negative answer:

Proposition 3.1 *Let Y be an arbitrary Banach space. Then*

$$(\mathfrak{A}_P^{*\Delta})^{inj}(P,Y) \subseteq \mathfrak{L}_2(P,Y),$$

and

$$\mathbf{L}_2(T) \leq (2\mathbf{C}_2(P') \cdot \mathbf{C}_2(P))^{\frac{3}{2}} \cdot (\mathbf{A}_P^{*\Delta})^{inj}(T)$$

for all operators $T \in (\mathfrak{A}_P^{*\Delta})^{inj}(P,Y)$.

Proof: First, let $F \in$ FIN and $T \in \mathfrak{L}(P,F)$ an arbitrary linear operator. Since the bidual $((\mathfrak{A}_P^{*\Delta})^{dd}, (\mathbf{A}_P^{*\Delta})^{dd}) =: (\mathfrak{B}_P, \mathbf{B}_P)$ always is left–accessible (!) and T is a finite rank operator, a copy of Pisier‘s construction immediately leads to

$$\mathbf{L}_2(T) \leq (2\mathbf{C}_2(P') \cdot \mathbf{C}_2(P))^{\frac{3}{2}} \cdot \mathbf{A}_P^{*\Delta}(T'') = (2\mathbf{C}_2(P') \cdot \mathbf{C}_2(P))^{\frac{3}{2}} \cdot \mathbf{B}_P(T).$$

Conjugation therefore implies

$$\mathfrak{D}_2(F,P) \overset{1}{=} \mathfrak{L}_2^{\Delta}(F,P) \subseteq \mathfrak{B}_P^{\Delta}(F,P) \overset{1}{\subseteq} \mathfrak{A}_P^*(F,P)$$

and

$$\mathbf{A}_P^*(S) \leq (2\mathbf{C}_2(P') \cdot \mathbf{C}_2(P))^{\frac{3}{2}} \cdot \mathbf{D}_2(S) \text{ for all } S \in \mathfrak{L}(F,P).$$

Since $(\mathfrak{A}_P^*, \mathbf{A}_P^*)$ is maximal, it even follows that

$$\mathfrak{D}_2(Y,P) \subseteq \mathfrak{A}_P^*(Y,P) \tag{1}$$

for all Banach spaces Y and

$$\mathbf{A}_P^*(S) \leq (2\mathbf{C}_2(P') \cdot \mathbf{C}_2(P))^{\frac{3}{2}} \cdot \mathbf{D}_2(S) \text{ for all } S \in \mathfrak{L}(Y,P).$$

Hence, conjugation of (1) finishes the proof.■

Note that the accessibility of the bidual $(\mathfrak{A}^{\Delta})^{dd}$ (which also was used in the previous proof) implies one of the main difficulties which appear repeatedly if one tries to construct a counterexample of a maximal Banach ideal $(\mathfrak{A}, \mathbf{A})$ so that the $\mathfrak{A} - LRP$ is not satisfied. In general, one is allowed to substitute statements related to properties of $\mathfrak{A}^{\Delta}$ through statements related to properties of the (left–)accessible bidual $(\mathfrak{A}^{\Delta})^{dd}$ so that these statements remain to be true, regardless whether the $\mathfrak{A} - LRP$ is satisfied or not! In particular, such statements cannot be used for a proof by contradiction. However, a first step towards a successful construction of such a candidate $(\mathfrak{A}, \mathbf{A})$ is given by the following factorization property for finite rank operators which had been introduced by Jarchow and Ott (cf. [7]). This factorization property not only turns out to be a useful tool in constructing such a counterexample; it even allows one to show that $\mathfrak{L}_\infty$ *is not totally accessible* – solving a problem of Defant and Floret (see [1, 21.12.] and [13, Theorem 4.1]). So, let us recall the definition of this factorization property and its implications:

Definition 3.2 (Jarchow/Ott) *Let $(\mathfrak{A}, \mathbf{A})$ and $(\mathfrak{B}, \mathbf{B})$ be arbitrary quasi–Banach ideals. Let $L \in \mathfrak{F}(X,Y)$ an arbitrary finite rank operator between two Banach spaces X and Y. Given $\epsilon > 0$, we can find a Banach space Z and operators $A \in \mathfrak{A}(Z,Y)$, $B \in \mathfrak{B}(X,Z)$ so that $L = AB$ and*

$$\mathbf{A}(A) \cdot \mathbf{B}(B) \leq (1+\epsilon) \cdot \mathbf{A} \circ \mathbf{B}(L).$$

(i) If the operator A is of finite rank, we say that $\mathfrak{A} \circ \mathfrak{B}$ has the property (I).

(ii) If the operator B is of finite rank, we say that $\mathfrak{A} \circ \mathfrak{B}$ has the property (S).

Important examples are the following (see [7, Lemma 2.4.]):

- If $\mathfrak{B}$ is injective, or if $\mathfrak{A}$ contains $\mathfrak{L}_2$ as a factor, then $\mathfrak{A} \circ \mathfrak{B}$ has the property (I).
- If $\mathfrak{A}$ is surjective, or if $\mathfrak{B}$ contains $\mathfrak{L}_2$ as a factor, then $\mathfrak{A} \circ \mathfrak{B}$ has the property (S).

Since $\mathfrak{L}_2 \circ \mathfrak{A}$ is injective for every quasi–Banach ideal $(\mathfrak{A}, \mathbf{A})$ (see [12, Lemma 5.1.]), $\mathfrak{B} \circ \mathfrak{L}_2 \circ \mathfrak{A}$ therefore has the property (I) as well as the property (S), for all quasi–Banach ideals $(\mathfrak{A}, \mathbf{A})$ and $(\mathfrak{B}, \mathbf{B})$. Such ideals are exactly those which contain $\mathfrak{L}_2$ as factor – in the sense of [7]. [13] explains in detail how the property (I) influences the structure of operator

ideals of type $\mathfrak{A}^{inj*} \stackrel{1}{=} \diagdown\mathfrak{A}^*$ and their conjugates, leading to another approach to construct a counterexample of a maximal Banach ideal with non–left–accessible conjugate. To this end, first note that for all Banach spaces X,Y and $X \stackrel{1}{\hookrightarrow} Z$, every operator $T \in (\mathfrak{A}^{inj})^*(X,Y) \stackrel{1}{=} \diagdown\mathfrak{A}^*(X,Y)$ satisfies the following extension property: given $\epsilon > 0$, there exists an operator $\widetilde{T} \in \diagdown\mathfrak{A}^*(Z,Y'')$ so that $j_Y T = \widetilde{T} J_X^Z$ and $\diagdown\mathbf{A}^*(\widetilde{T}) \leq (1+\epsilon)\cdot\diagdown\mathbf{A}^*(T)$ (see [6, Satz 7.14]). In particular, such an extension holds for all finite rank operators. However, we then cannot be sure that $\widetilde{T}$ is also as a *finite rank* operator. Here, property (I) comes into play – in the following sense:

Theorem 3.5 *Let $(\mathfrak{A}, \mathbf{A})$ be a Banach ideal so that $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ has the property (I). Let $\epsilon > 0$, X and Y be arbitrary Banach spaces and $L \in \mathfrak{F}(Y,X)$. Let Z be a Banach space which contains Y isometrically. Then there exists a finite rank operator $V \in \mathfrak{F}(Z, X'')$ so that $j_X L = V J_Y^Z$ and*

$$(\mathbf{A}^{inj})^*(V) \leq (1+\epsilon)\cdot(\mathbf{A}^{inj})^*(L).$$

If in addition the $\mathfrak{A}^* - LRP$ is satisfied, then V even can be chosen to be a finite rank operator with range in X and $L = V J_Y^Z$.

Consequently, this theorem leads to important implications which link the principle of local reflexivity for operator ideals with product ideals of type (I) such as the following ones (cf. [13]):

Theorem 3.6 *Let $(\mathfrak{A}, \mathbf{A})$ be a Banach ideal so that the $\mathfrak{A}^* - LRP$ is satisfied. Then*

$$\mathfrak{A}^{*\Delta inj} \stackrel{1}{\subseteq} (\mathfrak{A}^{*\Delta inj})^{dd}$$

If in addition, $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ has the property (I), then

$$(\mathfrak{A}^{*\Delta inj})^{dd} \stackrel{1}{=} \mathfrak{A}^{*\Delta inj} \stackrel{1}{=} \mathfrak{P}_1 \circ (\mathfrak{A}^*)^{-1} \stackrel{1}{=} \mathfrak{A}^{inj*\Delta} \stackrel{1}{=} (\mathfrak{A}^{inj*\Delta})^{dd}$$

Theorem 3.7 *Let $(\mathfrak{A}, \mathbf{A})$ be a left–accessible Banach ideal so that $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ has the property (I). Then $\mathfrak{A}^{inj}$ is totally accessible and $\mathfrak{A}^{inj} \stackrel{1}{\subseteq} (\mathfrak{A}^{inj})^{*\Delta}$. If in addition $(\mathfrak{A}, \mathbf{A})$ is maximal, then $(\mathfrak{A}^{inj})^*$ is also totally accessible.*

Theorem 3.8 *Let $(\mathfrak{A}, \mathbf{A})$ be a Banach ideal so that $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ has the property (I). If space$(\mathfrak{A})$ contains a Banach space X_0 so that X_0 has the bounded approximation property but X_0'' has not, then the $\mathfrak{A}^* - LRP$ cannot be satisfied.*

To construct such maximal Banach ideals, note again that $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ has the property (I), if $\mathfrak{A}^*$ contains $\mathfrak{L}_2$ as a factor. Since $\mathfrak{A}^*$ is a Banach ideal, we therefore have to look for maximal *Banach* ideals of type $\mathfrak{B} \circ \mathfrak{L}_2 \circ \mathfrak{C}$. A first investigation of geometrical properties of such product ideals was given in [12].

4 Normed operator ideal products

Unfortunately, we still cannot present explicite sufficient criteria which show the existence of (an equivalent) ideal norm on product ideals. It seems to be much more easier to show that a certain product ideal cannot be a normed one by using arguments which involve trace ideals and the ideal of nuclear operators (the smallest Banach ideal). Even more holds: if $\mathfrak{A} \circ \mathfrak{L}_2$ is a 1–Banach ideal for certain operator ideals $\mathfrak{A}$, then $\mathfrak{A} \circ \mathfrak{L}_2$ *is not* right–accessible (cf. Theorem 4.4)! However, let us look more carefully at such product ideals. First, we note an improvement of our own work (cf. [13, Theorem 4.2]):

Theorem 4.1 *Let* $(\mathfrak{A}, \mathbf{A})$ *be a maximal Banach ideal. Then both, the maximal* $\frac{1}{2}$*–Banach ideal* $\mathfrak{A}^{inj} \circ \mathfrak{L}_2$ *and the injective hull of the maximal* $\frac{1}{2}$*–Banach ideal* $\mathfrak{A} \circ \mathfrak{A}_2$ *are totally accessible.*

Proof: Since every Hilbert space has the metric approximation property and since $\mathfrak{A}^{inj} \stackrel{1}{=} (\mathfrak{A}^{inj})^{**}$ is right–accessible, an easy calculation shows that

$$\mathfrak{A}^{inj} \circ \mathfrak{L}_2 \stackrel{1}{=} (\mathfrak{A}^{inj})^{*\Delta} \circ \mathfrak{L}_2. \tag{2}$$

Since $(\mathfrak{A}^{inj})^{*\Delta}$ is right–accessible, the total accessibility of $\mathfrak{L}_2$ and the property (S) of the product ideal $(\mathfrak{A}^{inj})^{*\Delta} \circ \mathfrak{L}_2$ even imply that $(\mathfrak{A}^{inj})^{*\Delta} \circ \mathfrak{L}_2$ is totally accessible (cf. [13, Proposition 4.1]). Hence, $\mathfrak{A}^{inj} \circ \mathfrak{L}_2$ is totally accessible (due to (2)), and in particular we obtain that $(\mathfrak{A} \circ \mathfrak{L}_2)^{inj} \stackrel{1}{=} (\mathfrak{A}^{inj} \circ \mathfrak{L}_2)^{inj}$ is totally accessible.■

Now, let $(\mathfrak{A}, \mathbf{A})$ be a maximal Banach ideal so that $\mathbf{L}_2 \circ \mathbf{A}$ even is a norm on the (maximal) product ideal $(\mathfrak{L}_2 \circ \mathfrak{A}, \mathbf{L}_2 \circ \mathbf{A})$. Then $\mathfrak{A}^* \stackrel{1}{\subseteq} (\mathfrak{L}_2 \circ \mathfrak{A})^* \stackrel{1}{\subseteq} \mathfrak{L}_\infty$ (cf. [12, Proposition 5.1.]) and $\mathfrak{L}_\infty \stackrel{1}{=} \mathfrak{P}_1^\Delta \stackrel{1}{\subseteq} \mathfrak{N}^\Delta$. Given Banach spaces X and Y so that both, X' and Y have cotype 2, [15, Theorem 4.9.] tells us, that any finite rank operator $L \in \mathfrak{F}(Y, X)$ satisfies

$$\mathbf{N}(L) \leq (2\mathbf{C}_2(X') \cdot \mathbf{C}_2(Y))^{\frac{3}{2}} \cdot \mathbf{D}_2(L).$$

Hence,

$$\mathfrak{N}^\Delta(X, Y) \subseteq \mathfrak{D}_2^\Delta(X, Y) \stackrel{1}{=} \mathfrak{L}_2(X, Y),$$

and we have proven a rather surprising fact (revealing the strong influence of a *norm* on an operator ideal product):

Theorem 4.2 *Let* $(\mathfrak{A}, \mathbf{A})$ *be a maximal Banach ideal so that the product ideal* $(\mathfrak{L}_2 \circ \mathfrak{A}, \mathbf{L}_2 \circ \mathbf{A})$ *is normed. Let* X *and* Y *be arbitrary Banach spaces so that both,* X' *and* Y *have cotype 2. Then*

$$\mathfrak{A}^*(X, Y) \stackrel{1}{\subseteq} (\mathfrak{L}_2 \circ \mathfrak{A})^*(X, Y) \subseteq \mathfrak{L}_2(X, Y)$$

and

$$\mathbf{L}_2(T) \leq (2\mathbf{C}_2(X') \cdot \mathbf{C}_2(Y))^{\frac{3}{2}} \cdot (\mathbf{L}_2 \circ \mathbf{A})^*(T) \leq (2\mathbf{C}_2(X') \cdot \mathbf{C}_2(Y))^{\frac{3}{2}} \cdot \mathbf{A}^*(T)$$

for all operators $T \in \mathfrak{A}^*(X, Y)$.

To maintain the previous statement, even a permutation of the factors $\mathfrak{A}$ and $\mathfrak{L}_2$ in the product $\mathfrak{L}_2 \circ \mathfrak{A}$ is allowed:

Theorem 4.3 *Let $(\mathfrak{A}, \mathbf{A})$ be a maximal Banach ideal so that the product ideal $(\mathfrak{A}\circ\mathfrak{L}_2, \mathbf{A}\circ\mathbf{L}_2)$ is normed. Let X and Y be arbitrary Banach spaces so that both, X' and Y have cotype 2. Then*

$$\mathfrak{A}^*(X,Y) \overset{1}{\subseteq} (\mathfrak{A}\circ\mathfrak{L}_2)^*(X,Y) \subseteq \mathfrak{L}_2(X,Y)$$

and

$$\mathbf{L}_2(T) \leq (2\mathbf{C}_2(X')\cdot\mathbf{C}_2(Y))^{\frac{3}{2}} \cdot (\mathbf{A}\circ\mathbf{L}_2)^*(T) \leq (2\mathbf{C}_2(X')\cdot\mathbf{C}_2(Y))^{\frac{3}{2}} \cdot \mathbf{A}^*(T)$$

for all operators $T \in \mathfrak{A}^*(X,Y)$.

Proof: Let $(\mathfrak{A}, \mathbf{A})$ and X, Y be as before and let $\mathfrak{A}\circ\mathfrak{L}_2$ be normed. Then $\mathfrak{A} \overset{1}{=} \mathfrak{A}^{dd}$, and $\mathfrak{A}\circ\mathfrak{A}_2$ is a maximal (and therefore a regular) Banach ideal (cf. [13, Lemma 4.3]). Since the injective $\frac{1}{2}$–Banach ideal $\mathfrak{L}_2\circ\mathfrak{A}^d$ is also regular (cf. [12, Lemma 5.1]), an easy calculation shows that

$$(\mathfrak{A}\circ\mathfrak{L}_2)^d \overset{1}{=} \mathfrak{L}_2\circ\mathfrak{A}^d$$

and[2]

$$\mathfrak{A}\circ\mathfrak{L}_2 \overset{1}{=} (\mathfrak{L}_2\circ\mathfrak{A}^d)^d.$$

Since $\mathbf{A}\circ\mathbf{L}_2$ is a norm, $(\mathbf{A}\circ\mathbf{L}_2)^d$ obviously is a norm too. Hence, if we apply the previous theorem to the normed product ideal $(\mathfrak{A}\circ\mathfrak{L}_2)^d \overset{1}{=} \mathfrak{L}_2\circ\mathfrak{A}^d$, we obtain

$$\mathfrak{A}^{*d}(X,Y) \overset{1}{\subseteq} (\mathfrak{L}_2\circ\mathfrak{A}^d)^*(X,Y) \overset{1}{=} (\mathfrak{A}\circ\mathfrak{L}_2)^{*d}(X,Y) \subseteq \mathfrak{L}_2(X,Y),$$

and

$$\mathbf{L}_2(T) \leq C\cdot(\mathbf{A}\circ\mathbf{L}_2)^*(T') \leq C\cdot\mathbf{A}^*(T')$$

for *all* operators $T \in \mathfrak{A}^{*d}(X,Y)$ (where $C := (2\mathbf{C}_2(X')\cdot\mathbf{C}_2(Y))^{\frac{3}{2}}$). Now, since Y has the same coptype as its bidual $(Y')'$ with identical cotype constants (cf. [2, Corollary 11.9]), the proof is finished.■

Let $(\mathfrak{A}, \mathbf{A})$ be a given ultrastable quasi–Banach ideal so that $(\mathfrak{A}\circ\mathfrak{L}_2, \mathbf{A}\circ\mathbf{L}_2)$ is right–accessible. Our aim is to show that in this case $(\mathfrak{A}\circ\mathfrak{L}_2, \mathbf{A}\circ\mathbf{L}_2)$ and $(\mathfrak{L}_2\circ\mathfrak{A}^*, \mathbf{L}_2\circ\mathbf{A}^*)$ *both together* cannot be normed. To this end, we need a lemma which is of its own interest:

Lemma 4.1 *Let $(\mathfrak{A}_0, \mathbf{A}_0)$ be a maximal Banach ideal so that space$(\mathfrak{A}_0)$ contains a Banach space without the approximation property. Then there does not exist a maximal Banach ideal $(\mathfrak{C}, \mathbf{C})$ so that $\mathfrak{C}\circ\mathfrak{L}_\infty$ has the property (I) and $\mathfrak{C} \subseteq \mathfrak{A}_0^{-1}\circ\mathfrak{P}_1$.*

Proof: Assume that the statement is false. Then there exists a (maximal) Banach ideal $(\mathfrak{A}, \mathbf{A})$ so that $\mathfrak{A}_0 \subseteq \mathfrak{P}_1\circ(\mathfrak{A}^*)^{-1} \overset{1}{=} (\mathfrak{A}^{*\Delta})^{inj}$. Due to the assumed property (I) of $\mathfrak{A}^*\circ\mathfrak{L}_\infty$, the proof of Theorem 3.4 in [13] shows that even $((\mathfrak{A}^{*\Delta})^{inj})^{dd} \overset{1}{\subseteq} (\mathfrak{A}^{inj})^{*\Delta} \overset{1}{\subseteq} \mathfrak{N}^\Delta$. Since $\mathfrak{A}_0$ was assumed to be a maximal Banach ideal, we therefore obtain $\mathfrak{A}_0 \overset{1}{=} \mathfrak{A}_0^{dd} \overset{1}{\subseteq} \mathfrak{N}^\Delta$ which is a contradiction.■

Corollary 4.1 *Let $(\mathfrak{A}_0, \mathbf{A}_0)$ be a maximal Banach ideal so that space$(\mathfrak{A}_0)$ contains a Banach space without the approximation property. If $(\mathfrak{A}_0^{-1}\circ\mathfrak{P}_1)\circ\mathfrak{L}_\infty$ has the property (I), $\mathfrak{A}_0$ is not left–accessible.*

[2] In particular, it follows that $\mathfrak{A}\circ\mathfrak{L}_2$ is surjective (cf. [14, 8.5.9.]).

Theorem 4.4 *Let $(\mathfrak{B}, \mathbf{B})$ be an ultrastable quasi–Banach ideal so that $\mathfrak{B} \subseteq \mathfrak{L}_\infty$. If $\mathfrak{B} \circ \mathfrak{L}_2$ is right–accessible, $\mathfrak{B} \circ \mathfrak{L}_2$ cannot be a $1-$ Banach ideal.*

Proof: Assume that the statement is false and put $\mathfrak{B}_0 := (\mathfrak{L}_\infty \circ \mathfrak{L}_2)^*$ and $\mathfrak{A} := (\mathfrak{B} \circ \mathfrak{L}_2)^*$. Then

$$\mathfrak{A}^* \stackrel{1}{=} (\mathfrak{B} \circ \mathfrak{L}_2)^{\max} \stackrel{1}{=} (\mathfrak{B} \circ \mathfrak{L}_2)^{reg} \stackrel{1}{=} \mathfrak{B}^{reg} \circ \mathfrak{L}_2$$

is right–accessible (cf. [13, Proposition 2.3]) and contains $\mathfrak{L}_2$ as a factor so that in particular $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ has the property (I). Since $\mathfrak{B} \subseteq \mathfrak{L}_\infty$,

$$\mathfrak{B}_0 \circ \mathfrak{A}^* \subseteq \mathfrak{A} \circ \mathfrak{A}^* \stackrel{1}{\subseteq} \mathfrak{I} \stackrel{1}{\subseteq} \mathfrak{P}_1,$$

and it follows that $\mathfrak{A}^* \subseteq \mathfrak{B}_0^{-1} \circ \mathfrak{P}_1$. Since $Id_P \in \mathfrak{B}_0$ (cf. [13, Proposition 4.4]), Lemma 4.1 leads to a contradiction.$\blacksquare$

Now let us assume that $(\mathfrak{B}, \mathbf{B})$ is even is a *maximal Banach ideal* so that $\mathfrak{B} \subseteq \mathfrak{L}_\infty$. If $\mathfrak{B} \circ \mathfrak{L}_2$ were normed, then $\mathfrak{B} \circ \mathfrak{L}_2$ would be a maximal *and* surjective Banach ideal, implying that $\mathfrak{P}_1^d \stackrel{1}{=} \mathfrak{I}^{sur} \stackrel{1}{=} (\mathfrak{N}^{max})^{sur} \stackrel{1}{\subseteq} \mathfrak{B} \circ \mathfrak{L}_2$. Hence,

$$(\mathfrak{B} \circ \mathfrak{L}_2)^* \stackrel{1}{\subseteq} \mathfrak{P}_1^{d*} \stackrel{1}{=} \mathfrak{L}_1. \tag{3}$$

Since $\mathfrak{B} \subseteq \mathfrak{L}_\infty$, it even follows that $\mathfrak{P}_1 \stackrel{1}{=} \mathfrak{L}_\infty^* \subseteq (\mathfrak{B} \circ \mathfrak{L}_2)^* \stackrel{1}{\subseteq} \mathfrak{L}_1$ which is a contradiction (cf. [1, 27.2.]). So, in this case we obtain a stronger result:

Theorem 4.5 *Let $(\mathfrak{B}, \mathbf{B})$ be a maximal Banach ideal so that $\mathfrak{B} \subseteq \mathfrak{L}_\infty$. Then $\mathfrak{B} \circ \mathfrak{L}_2$ cannot be a $1-$ Banach ideal.*

We finish this survey paper with two statements linking the principle of local reflexivity with *normed* operator ideal products consisting of operators which factor through a Hilbert space (cf. [13, Theorem 4.4]):

Theorem 4.6 *Let $(\mathfrak{B}, \mathbf{B})$ be an ultrastable quasi–Banach ideal and X_0 be a Banach space without the bounded approximation property so that*

$$(\mathfrak{B} \circ \mathfrak{L}_2)^{reg}(\cdot, X_0) \stackrel{\mathfrak{F}}{\subseteq} \mathfrak{P}_1(\cdot, X_0)\ .$$

If $\mathfrak{B} \circ \mathfrak{L}_2$ is a 1–Banach ideal, then the $(\mathfrak{B} \circ \mathfrak{L}_2)^{**} - LRP$ cannot be satisfied.

Theorem 4.7 *Let $(\mathfrak{B}, \mathbf{B})$ be a maximal Banach ideal so that $\mathfrak{P}_1 \circ \mathfrak{B}^{-1}$ contains $\mathfrak{L}_2$ as a factor and $\mathfrak{L}_\infty \circ (\mathfrak{P}_1 \circ \mathfrak{B}^{-1})$ is normed. If there exists a Banach space X_0 without the approximation property so that $X_0 \in space(\mathfrak{B})$, then the $\mathfrak{B} - LRP$ is false.*

Proof: Assume that the statement is not true. Then $\mathfrak{A}_0 := \mathfrak{P}_1 \circ \mathfrak{B}^{-1} \stackrel{1}{=} (\mathfrak{B}^\Delta)^{inj}$ is totally accessible. Since $\mathfrak{A}_0$ contains $\mathfrak{L}_2$ as a factor, $\mathfrak{L}_\infty \circ \mathfrak{A}_0$ in particular has the property (S), and it follows that $\mathfrak{L}_\infty \circ \mathfrak{A}_0$ even is totally accessible (cf. [13, Proposition 4.1]). Hence,

$$\mathfrak{B} \stackrel{1}{\subseteq} \mathfrak{A}_0^{-1} \circ \mathfrak{P}_1 \stackrel{1}{=} (\mathfrak{L}_\infty \circ \mathfrak{A}_0)^* \stackrel{1}{=} (\mathfrak{L}_\infty \circ \mathfrak{A}_0)^\Delta \stackrel{1}{\subseteq} \mathfrak{N}^\Delta,$$

and we obtain a contradiction.$\blacksquare$

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